lec 24

THE Fundamental Thm of Calculus

- Establishes a connection between differential calculus and integral calculus.

Differential calculus : Arose from tangent problems
Integral calculus : Arose from area problems.

$\Big\}$ Turns out these are inverses.

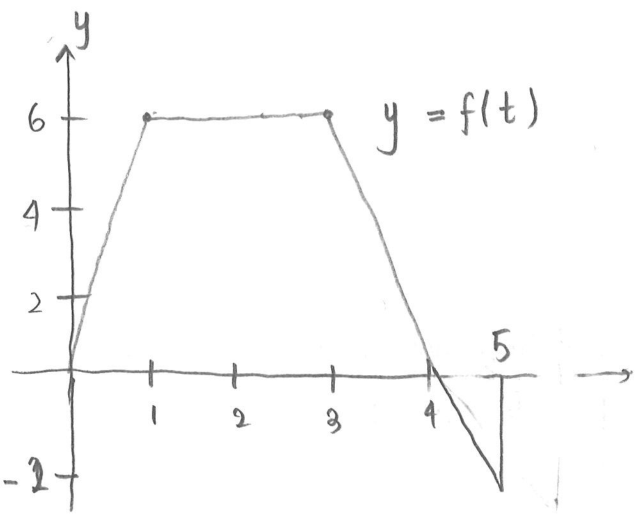
- The F.T.C gives the precise inverse relationship between derivative and the integral.

- The first part of F.T. deals with functions defined by an equation of the form

- $g(x) = \int_a^x f(t)\,dt$ where f is a continuous on $[a, b]$ and x-varies between a and b.

- g depends on x, which appears as an upper limit in the integral.

Ex



If f is the function whose graph is as shown &

$g(x) = \int_0^x f(t)\,dt$. Find values $g(0), g(1), g(2), g(3), g(4), g(5)$

Soln

$g(0) = \int_0^0 f(t)\,dt = 0$

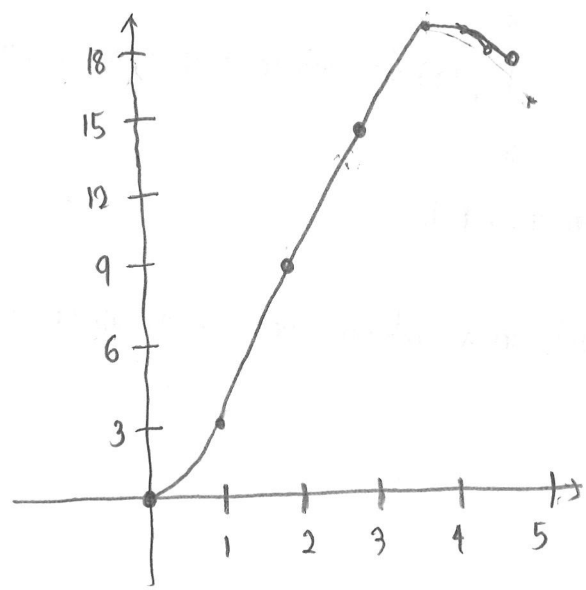
$g(1) = \int_0^1 f(t)\,dt = 3$

$g(2) = \int_0^2 f(t)\,dt = 3+6 = 9$

$g(3) = \int_0^3 f(t)\,dt = 3+12 = 15$

$g(4) = \int_0^4 f(t)\,dt = 15+3 = 18$

$g(5) = \int_0^5 f(t)\,dt = 18 - 1 = 17$

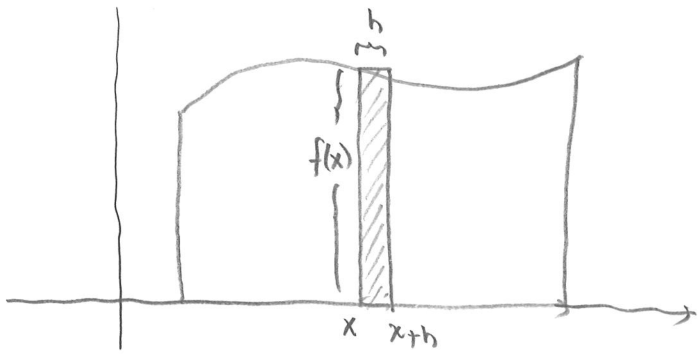


IDEA BEHIND FTC

let's consider a positive function

$$g(x) = \int_a^x f(t)\,dt$$

$$g(x+h) = \int_a^{x+h} f(t)\,dt$$



Then, $g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt = \int_x^{x+h} f(t)\,dt$

Then, $\dfrac{g(x+h) - g(x)}{h} \approx \dfrac{f(x).h}{h} \approx f(x)$

So app, $g'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} \approx f(x)$

Ex If $g(x) = \int_a^x f(t)\,dt$,

$a = 1$, $f(t) = t^2$. Find a formula for $g(x)$ and calculate $g'(x)$.

Soln $$g(x) = \int_1^x t^2\,dt \overset{\text{ev. thm}}{=} \left[\frac{t^3}{3}\right]_1^x = \frac{x^3}{3} - \frac{1}{3} = \frac{x^3-1}{3}$$

Then, $g'(x) = x^2 = f(x)$.

This is not a random occurence

FTC Part 1

If f is continuous on $[a,b]$, then g defined by

$$g(x) = \int_a^x f(t), \quad a \le x \le b,$$ is an antiderivative of f i.e.

$g'(x) = f(x)$ for $a < x < b$. $\boxtimes$

Ex Find the derivative of the function

$$g(x) = \int_0^x \sqrt{1+t^2}\ dt$$

Since $f(t) = \sqrt{1+t^2}$ is continuous, by FTC part 1,

$$g'(x) = \sqrt{1+x^2}$$

Ex Find $\frac{d}{dx}\int_1^{x^4} \sec t \, dt$

Ans Note that the upper limit is x^4, so we have to use Chain Rule along with FTC part 1.

Then let $u = x^4$

$$\frac{d}{dx}\int_1^{x^4} \sec t \, dt = \frac{d}{dx}\int_1^{u} \sec t \, dt = \frac{d}{du}\left[\int_1^{u} \sec t \, dt\right] \cdot \frac{du}{dx}$$

$$= \sec u \cdot \frac{du}{dx} = \sec(x^4) \cdot 4x^3$$

Differentiation and integration as inverse processes

The Fundamental Thm of Calculus

Suppose f is continuous on $[a,b]$

1) If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$

2) $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f i.e $F' = f$.

Remark

Part 2 is just the evaluation Thm

Last class we reformulated Part 2 as the Net Change Thm,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

(This can be thought of as differentiating F and integrating the result, we arrive back at original function F but in the form $F(b) - F(a)$)

Remark

Part 1 can be rewritten as $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

which says that if f is integrated and the result is then differentiated, we arrive back at the original function f.

AVERAGE VALUE OF A FUNCTION

Ex $\{a_1, a_2, \ldots, a_n\}$, then average $\bar{a} = \dfrac{a_1 + \cdots + a_n}{n}$

But how do you compute the average temperature during a day if infinitely many temperature reading are possible?

Let $y = f(x)$, $a \le x \le b$.

Then the average value of f on the interval $[a, b]$ as

$$\boxed{f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx}$$

Ex Find the average value of the function on the interval

$$f(\theta) = \sec\theta . \tan\theta \;,\quad \left[0, \frac{\pi}{4}\right].$$

Sol $a = 0$, $b = \frac{\pi}{4}$

$$f_{ave} = \frac{1}{\frac{\pi}{4} - 0} \int_0^{\pi/4} \sec\theta \tan\theta \, d\theta = \frac{1}{\frac{\pi}{4}} \Big[\sec\theta \Big]_0^{\frac{\pi}{4}}$$

$$= \frac{4}{\pi}\left(\sec\frac{\pi}{4} - \sec 0 \right) = \frac{4}{\pi}\left(\sqrt{2} - 1 \right)$$

If $T(t)$ is the temperature at time t, we might wonder if there is a specific time when the temperature is the same as the average temperature.

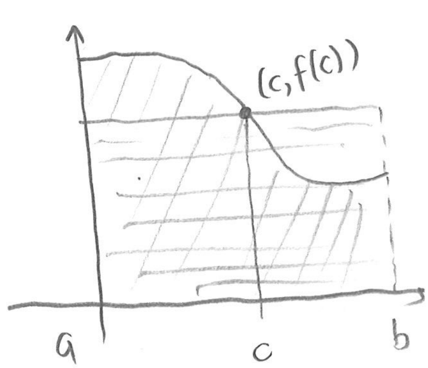
The following thm says that this is true for a continuous function.

THE MEAN VALUE THM FOR INTEGRALS

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{i.e.} \quad \boxed{\int_a^b f(x)\,dx = f(c)(b-a)}$$

The geometric interpretation of the mean value Thm is that for a positive functions f, there a number c such that the rectangle with base $[a,b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.



Random Questions

1) Find $\dfrac{d}{dy} \displaystyle\int_y^1 \sqrt[3]{1+t^2}\, dt$

$$= \frac{d}{dy}\left[-\int_1^y \sqrt[3]{1+t^2}\, dt\right] = -\sqrt[3]{1+y^2}$$

2) Find $\dfrac{d}{dx} \displaystyle\int_0^{\sqrt{x}} (1+t)^9\, dt$, let $u=\sqrt{x}$

$$= \frac{d}{dx}\left[\int_0^{\sqrt{x}} (1+t)^9\, dt\right] = \frac{d}{du}\int_0^u (1+t)^9\, dt \cdot \frac{du}{dx}$$

$$= (1+u)^9 \cdot \frac{1}{2}x^{-1/2} = (1+\sqrt{x})^9 \cdot \frac{1}{2\sqrt{x}}$$